Game Theory

Pleasanton Math Circle: Middle School

Rohan Garg, Ryan Vir

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§1 Easier Games

Problem 1.1. Alice and Bob start with the number 0. Starting with Alice, each player can increment the number by either $1, 2, 3$. The person who says the first number greater than or equal to 21 loses. Can any of the players guarantee victory? If yes, what is the strategy they follow?

Proof. Bob can guarantee victory by making it a multiple of 4 on his turn every time. Then Bob will say 20, and Alice loses.

Problem 1.2. Try the same thing but now, the first person who says a number greater than or equal to 22 loses.

Proof. Alice says 1. Now on Alice's turns, she always says the number that has a remainder of 1 divided by 4. Eventually, she will say 21, and Bob loses.

Problem 1.3. Generalize this game. If we can increment by $1, 2, \ldots, a$ and the first person to say a number greater than or equal to b loses, who wins? Your answer should have a few cases.

Proof. If b has a remainder of 1 when divided by $a + 1$, then player 2 wins. If not, then player 1 wins.

§2 Classic Heap Game

Now imagine we have two piles of coins. Each pile has some number of coins, m, n. On a move, a player can remove any number of coins from either pile. A player wins if they take the last coin. Let's work through some cases. (There are two players and they alternate turns)

Problem 2.1. Which player wins if there is 1 coin in each pile? What about 2? 3? 4? Can you prove your observations more generally?

Proof. Player 2 can copy what player 1 does in the other pile, forcing player 2 to remove the last coin and henceforth, player 2 wins.

Problem 2.2. Using your previous observations, can you prove who wins for any m, n?

Proof. If $m = n$, then player 2 wins as we saw above. If $m \neq n$, then on the first turn, Player 1 can balance the piles. Then, Player 1 uses the same strategy player 2 used above and can win.

§3 Wythoff's Game

Wythoff's game is a classic spinoff of the heap game. We have the same setup, but we can take coins from both piles this time. The only catch is that if we take coins from both piles, we must take the same amount from each pile. Before we start, there is some important notation we must consider. We denote a game with m coins in the first pile and n coins in the second as (m, n).

Definition 3.1 (Winning and Losing Positions)**.** A winning position is a position such that the active player has a strategy that leads to victory. Similarly, a losing position is a position such that the active player loses in every case.

Problem 3.2. Given that (m, n) is a losing position, find all winning positions only using this information.

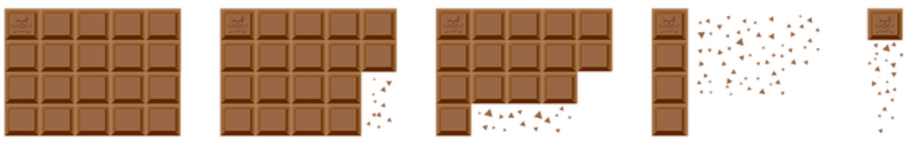
Proof. $(m + k, n)$, $(m, n + k)$, and $(m + k, n + k)$ are all winning positions for all integers k.

Problem 3.3. Using this information, find all losing positions with $m \leq 10, n \leq 10$. Graph these points and try to find a pattern between these values. If you don't see one, find some more losing positions. Form some conjectures and ideas about possible values of losing positions. The proof is very challenging, so we won't be going over it.

Proof. The losing positions are $(0, 0), (1, 2), (3, 5), (4, 7), (6, 10)$ and their reverses (i.e. the reverse of $(1, 2)$ is $(2, 1)$). We see that their graph seems to be two lines, one with slope ϕ and the other with slope $\frac{1}{\phi}$ (ϕ is $\frac{1+\sqrt{5}}{2}$).

§4 Chomp!

Chomp is played on a $m \times n$ grid, where in a move, a player chooses a unit square on the grid and then removes (or bites off) all the unit squares to the right or below that unit square. An example of possible moves on a 5×4 grid are shown below:



The player who eats the last piece loses. Let's explore this game in a bit more depth.

Problem 4.1. Try to find a winning strategy for the first player on a $n \times n$ grid (except when $n = 1$).

Proof. The first player chooses the square one right and one down to the top left corner and removes a $n - 1 \times n - 1$ square portion of the grid. Now, the first player copies what the second player does until the second player has no option but to eat the last piece.

§4.1 Strategy Stealing Arguments

This is a way to prove that the first player always wins a game. The format of the proof works as such:

- Assume the other player can win on some setup (this is a losing position, since it is currently the first player's turn).
- Choose a move that the first player makes (this is the hardest part) such that it is easy to do the 4th bullet point.

- We are now at a winning position, so there is a move the second player can make to give the first player a losing position.
- Show that the first player could have made this same move on their turn. This contradicts the fact that we started at a losing position, meaning the second player doesn't have a winning strategy.

Problem 4.2. Use a Strategy Stealing Argument to prove that the first player wins no matter what m, n are (except $m = n = 1$).

Proof. Assume the second player has a winning strategy. The first player removes the bottom right corner. Now since the second player has a strategy, they choose a square that is a winning position for them. But on the first player's move, they could have chosen the same square, and hence, removing the same squares. This means the first player can steal the second player's strategy and win instead.

> **Remark 4.3.** Finding the exact strategy that the first player follows for an arbitrary grid is an open problem, yet to be solved.

§5 Bonus Section

§5.1 Bases

You might think this isn't related, but you'll see how this ties into game theory in a bit! We write numbers in base 10, because a string of digits, say 632 for example, is treated as $6 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0$. Similarly, we can express in a number in base 2. For example, we can write 40 in base 2 as 101000_2, since this is $2^5 \cdot 1 + 2^4 \cdot 0 + 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 0 + 2^0 \cdot 0 = 40$.

§5.2 Nim-Sum

Given a list of numbers, the nim sum is defined as follows:

- Convert all the numbers to base 2.
- Consider each digit position. If an even number of the numbers have a 1 in that digit spot, do nothing. If an odd number of the numbers have a 1 in that spot, then add that power of 2 to the nim-sum.

Here is an example. If we have $13, 19, 33$, then the base 2 representations are $1101, 10011, 100001$. We see that each of the digits from $1 - 6$ appear an odd number of times, so the nim-sum is $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$.

§5.3 Connecting Bases with Game Theory

Consider the game from section 2. Instead of two piles, we can have any number of piles.

Problem 5.1. We already found all the winning and losing positions for the 2 pile version. Consider the nim sum of the winning cases and the losing cases. What do you notice?

Proof. The nim sum when the second player won was always 0.

Problem 5.2. Find some winning setups and some losing setups when there are 3 piles. Do the same thing and compute the nim sum. Do you see a pattern?

Proof. We see the same thing here. The nim sum when the second player wins is always 0.

Problem 5.3. Challenge: If you are familiar with induction, try to prove your observations.

Proof. We can use induction. The details are grueling, but here is the general idea:

- Show that if the nim sum of some piles was 0, then after making a move, it has to become nonzero.
- Show that there exists some move such that any piles with a nonzero nim sum will have a nim sum of zero after the move.